The knapsack problem

Given a knapsack with maximum capacity $W$, and a set consisting of $n$ items. Each item $i$ has some weight $w_i$ and benefit value $b_i$ (all $w_i$, $b_i$ and $W$ are integer values)
Problem: How to pack the knapsack to achieve maximum total value of packed items?

Let $x_i$ be a variable for the item $i$. $x_i=1$ if the items is taken in the knapsack, else it is equal to 0. The problem is the following:

$$\begin{align*}
\text{max} & \sum_{i=1}^{n} x_i b_i \\
\text{s.t} & \sum_{i=1}^{n} x_i w_i \leq W \\
& x_i \in [0,1]
\end{align*}$$

Solve the following instance of the knapsack problem with four items where $W=10$.

<table>
<thead>
<tr>
<th>i</th>
<th>$b_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_i$</td>
<td></td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

For example, {1,3} have a total weight of 10 which is legit. The total value is 25+20=45.

Question 1: Greedy

George Dantzig proposed a greedy algorithm to solve the knapsack problem. His version sorts the items in decreasing order of value per unit of weight. It then proceeds to insert them into the knapsack, starting with the first kind of item until there is no longer space in the sack for more. Use this algorithm for the previous problem.
**Question 2: Dynamic programming**

Use the following dynamic program on the previous problem.
Define $m[i,w]$ to be the maximum value that can be attained with weight less than or equal to $w$ using items up to $i$ (first $i$ items). For example, $m[2,5]$ is the optimal value of the knapsack of weight 5 using only item 1 and 2.

```
for j from 0 to W do:
    m[0, j] := 0

for k from 0 to n do:
    m[k, 0] := 0

for i from 1 to n do:
    for j from 0 to W do:
        if $w[i] > j$ then:
            $m[i, j] := m[i-1, j]$
        else:
            $m[i, j] := \max(m[i-1, j], m[i-1, j-w[i]] + b[i])$
```

Write the mathematical system, explain each line.

```
\begin{align*}
    \text{initialize ...} \\
    \text{if ...} \\
    \text{else if ...} \\
    \text{etc.}
\end{align*}
```

**Question 3: Process**

Write the table. Where is the optimal value? What is the optimal value?

**Question 4: How to find which items are used**

In the dynamic program, how can we find which items are used to reach the maximum value? Explain with our own words and with the table.

**The more you know**

*The knapsack problem is the 4th most used problem. For example, the knapsack problem is frequently used to define the best way to consume a defined amount of energy.*
Question 1

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>b_i</td>
<td>25</td>
<td>15</td>
<td>20</td>
<td>36</td>
</tr>
<tr>
<td>w_i</td>
<td>7</td>
<td>2</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>

Item 1: $25/7= 3.57$

Item 2: $15/2= 7.5$

Item 3: $20/3= 6.67$

Item4: $36/6 =6$

At first, we put item2 in the knapsack, we still have 8 unused weight. Then we put item3, still have 5. We can’t put item4 neither item 1.

Total value = 15+20=35.

Question 2

Initialize: there is no item or no space in the knapsack, the optimal value is zero.

Process: the optimal value is the maximum value without the new item $m[i-1, w]$ or with the item considering we have the weight to put it into the knapsack, i.e. $m[i-1, w-w_i]+b_i$.

Question 3

<table>
<thead>
<tr>
<th>W / i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>0</td>
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<td>0</td>
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</tr>
<tr>
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<td>0</td>
<td>15</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>15</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
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<td>15</td>
<td>35</td>
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</tr>
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<td>0</td>
<td>15</td>
<td>35</td>
<td>36</td>
</tr>
<tr>
<td>6</td>
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<td>0</td>
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<td>25</td>
<td>36</td>
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<td>25</td>
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<td>25</td>
<td>25</td>
<td>35</td>
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<tr>
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<td>25</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>25</td>
<td>40</td>
<td>56</td>
</tr>
</tbody>
</table>
**Question 4**
Reverse the mathematical system to know from where you get the optimal value at $m[n,W]$.

$m[i,w]=m[i-1,w]$ or $m[i-1,w-w_i]+b_i$.

$m[4,10]=56$ comes from $m[3,4]=20$ more $b_4=36$. So the item 4 is in the bag.

$m[3,4]=20$ comes from $m[1,4]=0$ more $b_3=20$. So the item 3 is in the bag.

$M[1,4]=0$ means no item in this bag, END.