Largest Inscribed Ball and Minimal Enclosing Box for Convex Maximization Problems

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Outline

Definition

A problem is a convex minimization if the following conditions are true:

- It is a minimization;
- Objectif function is convex;
- Domain is convex.

If one of these conditions is violated the problem is a non-convex minimization.
If objectif function is concave or 'max' instead of 'min' then the problem is called convex maximization.

Pardalos and Rosen (1987), Floudas and Pardalos (1992), Horst and Tuy (1993), Hendrix and Toth (2010)

An important problem

Convex maximization problems are used in many applications:

1. Economics and engineering design
   1. Fixed charge problem;
   2. Bid evaluation problem.
2. Problems that can be transformed
   1. Integer programming;
   2. Bilinear programming;
   3. Complementarity problems;
   4. Mutliplicative programs.

Horst and Tuy (1993)

Problematic: applications in Smart Grid

How to find a global maximum

$$\begin{cases} \text{maximize} \;\; f(x), \\ \text{subject to } x \in D \end{cases} \quad (CM)$$

where D denote a bounded and full dimensional polyhedron (called polytope).

Some classical methods:

- Vertex enumeration or outer approximation: approximation with a known polytope (a global maximum is obtained on a vertex), number of vertices of D are fairly large.
- Facet enumeration or inner approximation: same as vertex enumeration.
- Polyhedral partitions: to split the domain into a finite number of subpolyhedra.
- Upper level sets: hard to escape from a local maximum.

Problem

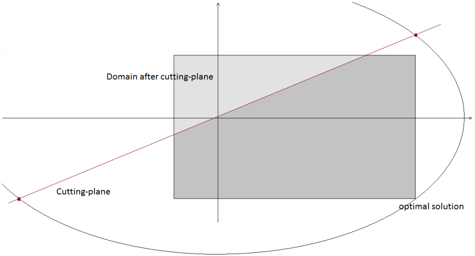
How to escape to a local solution area?
In Smart Grid applications, the quality of a local maximum plays an important role.

Cutting-plane algorithm (Tuy 1964)

One cuts off a part of D, where values of function $f(.)$ are less or equal than $f(y)$.
Let y be a known vertex of D:

1. Find points y^i on level set $f(y^i) = f(y)$ following n edges from y.
2. Construct the hyperplane $\{x \mid \langle c, x\rangle = \gamma\}$ passing through these points.
3. The constraint $\langle c, x\rangle \geq \gamma$ doesn't change the nature of the problem:

$$\begin{cases} \text{maximize } \; f(x), \\ \text{subject to } x \in D, \langle c, x\rangle \geq \gamma \end{cases} \quad (CM)$$

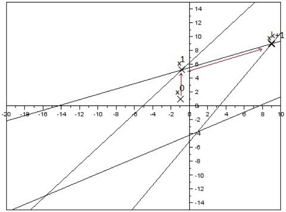


Local search

Necessary optimality condition for a local solution $\partial f(y) \cap N(D,y) \neq \emptyset$ where $\partial f(.)$ stand for the sub-differential and $N(.,.)$ the normal cone.
Local search is done iteratively by:

$$x^{k+1} = argmax \left\{ \left\langle \nabla f(x^k), x \right\rangle \mid x \in D \right\}$$

with given $x^0 \in D$.
If $x^k = x^{k+1}$ then local optimality condition is satisfied at x^k.



$$\begin{cases} max & x_1^2 + x_2^2 \\ s.t. & -5x_1 + 13x_2 \leq 72 \\ & 11x_1 - 7x_2 \leq 36 \\ & 5x_1 - 9x_2 \leq 28 \\ & -11x_1 + 9x_2 \leq 56 \end{cases} \qquad \text{(P6)}$$

A classic problem

In our industrial application, we need to solve the simplest version of convex maximization problem:

A classic problem

$$\left\{ \begin{array}{l} \text{maximize} \quad \|x\|^2, \\ \text{subject to } x \in D \end{array} \right. \quad (NM)$$

where D denote a bounded and full dimensional polyhedron (called polytope), and $\|.\|$ stands for the Euclidean norm.

Largest inscribed ball

Why the largest inscribed ball is helpful for convex maximization ?

Motivation

- Problem of finding the largest inscribed ball is very easy to solve.
- There exists an extreme point of D which globally maximizes the problem (NM).
- There is no extreme point of D belonging to an inscribed ball S in D.

$$\begin{cases} \text{maximize} & \|x\|^2, \\ \text{subject to } x \in D \end{cases} \iff \begin{cases} \text{maximize} & \|x\|^2, \\ \text{subject to } x \in D \setminus S \end{cases}$$

Largest inscribed ball

How to find the radius of the largest inscribed ball:

1. Let $\delta(x)$ be the radius of the largest inscribed ball centered at $x \in int(D)$
   1. Let $r_i(x)$ be the distance from x to the constraint i, $r_i(x) = \frac{-\langle a^i, x\rangle + b_i}{\|a^i\|^2}$
   2. Therefore $\delta(x) = min\{r_i(x) : i = 1 \to m\}$
2. $\delta(x) = min\{-\langle a^i, x\rangle + b_i : i = 1 \to m\}$ if constraints are normalized $\|a^i\| = 1$
3. $\delta(x)$ is a concave function
4. $x \in D$ iff $\delta(x) \geq 0$.

Largest inscribed ball

The following problem

$$\begin{cases} \text{maximize} \ \ \delta(x), \\ \text{subject to } \delta(x) \leq -\langle a^i, x\rangle + b_i \ : \ i = 1 \to m \end{cases} \quad (SLP)$$

can be solved by solving the following linear program in $\mathbb{R}^{n+1}$:

$$\begin{cases} \text{maximize} \ \ x_{n+1}, \\ \text{subject to } \langle a^i, x\rangle + x_{n+1} \leq b_i \ : \ i = 1 \to m \end{cases} \quad (SLP)$$

Maximum over a ball

Let $w \in \mathbb{R}^n$ be a center of a ball, $r > 0$ be its radius, then a closed ball is defined by:

$$S = \left\{ x \in \mathbb{R}^n \mid \|x - w\|^2 \leq r^2 \right\}.$$

Consider the following problem:

$$\begin{cases} \text{maximize} & \|x\|^2, \\ \text{subject to} & \|x - w\|^2 \leq r^2 \end{cases} \quad (SNM)$$

By KKT condition, an optimal solution u is on the boundary of the ball such as $\exists \alpha \geq 1,\ u = \alpha w$.
Point $u = \left(1 + \frac{r}{\|w\|}\right) w$ solves (SNM).

Smallest enclosing box

How to define the smallest outer box B:

1. For each constraints i:
   1. $U_i = argmax \left\{\langle e^i, x\rangle \mid x \in D\right\}$
   2. $L_i = argmin \left\{\langle e^i, x\rangle \mid x \in D\right\}$

Future works: From box to hyper-rectangle (not following the identity matrix) thanks to the Gram-Schmidt process applicated to a facet.

Maximum over a box

Let $L, U \in \mathbb{R}^n$ be two vectors such that $L_i < U_i$. The box is defined by:

$$B = \{x \in \mathbb{R}^n \mid L_i \leq x_i \leq U_i, i = 1 \to n\}.$$

Consider the following problem:

$$\begin{cases} \text{maximize } \ \|x\|^2, \\ \text{subject to } L_i \leq x_i \leq U_i, i = 1 \to n \end{cases} \quad (BNM)$$

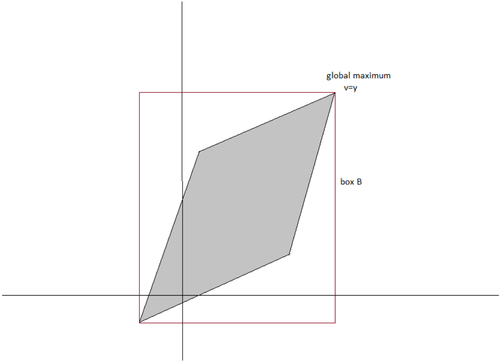
The problem is separable in x_i and the solution of the problem is v with $v_i = \left(\ max\{| L_i |,| U_i |\} \ \right)$, $i = 1 \to n$.

Same solutions over the box and over the domain

Lower and upper bounds: $\max_{x \in S} f(x) < \max_{x \in D} f(x) \leq \max_{x \in B} f(x)$

The inclusion $D \subset B$ implies:

1. $f(v) \geq f(x), x \in D$
2. If $y = v$ then y is the global maximum of (NM).



Empty domain after cutting-plane

1. Let $\langle c, x\rangle = \gamma$ be the cutting-plane, and $\mathscr{L}_f(\alpha)$ be the Lebesgue set of f on α, $\mathscr{L}_f(\alpha) = \{x \mid f(x) \leq \alpha\}$.
2. Point y is a global optimum of (NM) iff $D \subset \mathscr{L}_f(f(y))$.
3. If the domain after cutting plane is empty, $\{x \in D \mid \langle c, x\rangle > \gamma\} = \emptyset$, then $\forall x \in D, \langle c, x\rangle < \gamma$.
4. Point y is a global maximum of (NM).

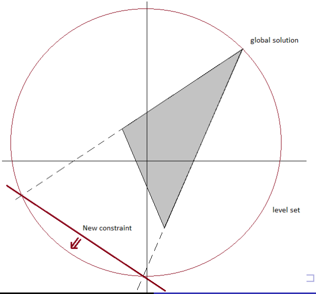
The domain after cutting-plane could be non-empty even when y is a global maximum.

Empty domain after cutting-plane

Let $\hat{x}$ be the solution of the following problem:

$$\begin{cases} \text{maximize} & \langle c, x \rangle, \\ \text{subject to } x \in D \end{cases}$$

1. If $\gamma < \langle c, \hat{x} \rangle$ then there exists $x \in D$ such that $\gamma \leq \langle c, x \rangle \leq \langle c, \hat{x} \rangle$. The domain is nonempty.
2. If $\gamma > \langle c, \hat{x} \rangle$, so $\langle c, x \rangle \leq \langle c, \hat{x} \rangle \leq \gamma$ and $\nexists x \in D$ such that $\gamma \leq \langle c, x \rangle$. The domain is empty.

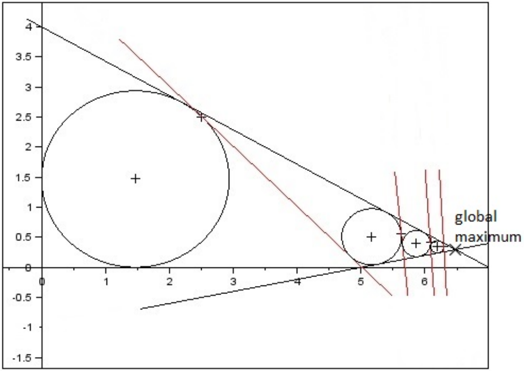


Algorithm 1

Algorithm 1: Inner approximation: IA

Data: $Ax \leq b$
Result: Global optimum of (NM)
Initialization : $r_{min} = \epsilon$ and $condition = false$;
while $condition = false$ **do**
    while $r > r_{min}$ **do**
        Find r, w the solution of (SLP);
        Find u the global optimum of (SNM);
        Add a temporary constraint to (NM): $\langle w, x \rangle \geq \|w\|^2 + r^2$;
    end
    Remove all temporary constraints;
    Local research starting from u;
    if *global optimality conditions checked* **then**
        $condition = true$;
        Keep in memory the optimum;
    else
        Construct cutting plane at the local optimum;
        Keep in memory the optimum;
    end
end
Compare all optima in memory;
Return the global maximum;

Example 1



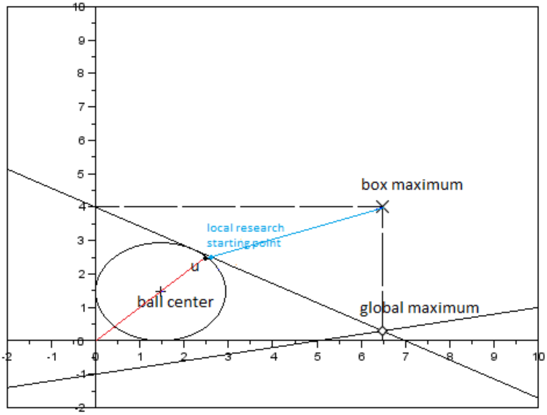
(P4)

Algorithm 2

Algorithm 2: Inner and outer approximation: IOA

Data: $Ax \leq b$
Result: Global optimum of (NM)
Initialization : $condition = false$;
while $condition = false$ **do**
    Resolve (SLP) ;
    Find u the global optimum of (SNM) ;
    Find v the global optimum (BNM) ;
    Find the intersection of interval $[uv]$ with boundary of D;
    Local search starting from the intersection point;
    if *global optimality conditions checked* **then**
        $condition = true$;
        Keep in memory the optimum;
    else
        Construct cutting plane at the local optimum;
        Keep in memory the optimum;
    end
end
Compare all optima in memory;
Return the global maximum;

Example 2


box maximum
local research
starting point
u
ball center
global maximum

(P4)

Results

problem	n	iIA	iIOA	bkv	optimal value	tIA	tIOA
TP2.1	5	3	3	-17	-17	0.1	0.2
TP2.6	10	2	4	-39	-39	0.3	0.7
TP2.7.1	20	3	2	-394.7506	-394.7506	1.5	1.0
TP2.7.3	20	3	2	-8695.01193	-8695.01193	1.5	1.0
P4	2	1	1	42.0976	42.0976	0.1	0.1
P6	2	2	1	162	162	0.1	0.1
P11	100	2	1	1541089	1541089	6	2.5
P11	200	2	1	4150.41013	4150.41013	35	14.7

Computer: IntelCore2 Duo processor, 3.16GHz CPU and 4GB of RAM.

Software: Scilab with Linpro.

Problems P#: *Enkhbat, R.: An algorithm for maximizing a convex function over a simple set. Journal of Global Optimization 8(4), 379-391 (1996)*

Problems TP#: *Floudas, C.A., Pardalos, P.M.: A collection of test problems for constrained global optimization algorithms, Lecture Notes in Computer Science, vol. 455. Springer-Verlag, Berlin (1990)*